

EXAMPLES OF FINITELY DETERMINED MAP-GERMS OF CORANK 3 SUPPORTING MOND'S $\mu \geq \tau$ -TYPE CONJECTURE

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Dedicated to my parents

1. INTRODUCTION

A famous $\mu \geq \tau$ -type conjecture by D. Mond states that for a finitely \mathcal{A} -determined map-germ from \mathbb{C}^n to \mathbb{C}^{n+1} , provided $(n, n+1)$ is in the range of Mather's nice dimensions,

$$(1) \quad \mu_I \geq \mathcal{A}_e\text{-codimension},$$

and with equality if the map-germ is weighted homogeneous. The conjecture was proven for $n = 1$ by Mond ([18]) and for $n = 2$ by Pellikaan and de Jong (unpublished), de Jong and Straten ([5]) and D. Mond ([17]). It is still open for $n \geq 3$. Several examples supporting the conjecture were given in the case of map-germs of corank 1 ([13]) and corank 2 ([1]). It was believed by some that it would only hold for map-germs of corank ≤ 2 . In this article, we provide examples of finitely determined map-germs of corank 3 defined from \mathbb{C}^3 to \mathbb{C}^4 which satisfy the conjecture (Section 3). These are the first examples in the literature known to the author.

2. TERMINOLOGY AND NOTATIONS

2.1. Finite determinacy. Our terminology is standard, but the details can be found in [24] or [14]. We denote the space of holomorphic map-germs $f: (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$ by $\mathcal{E}_{n,p}^0$. The group $\mathcal{A} := \text{Diff}(\mathbb{C}^n, 0) \times \text{Diff}(\mathbb{C}^p, 0)$ of local diffeomorphisms acts on $\mathcal{E}_{n,p}^0$ by

$$(\phi, \psi) \cdot f \mapsto \psi \circ f \circ \phi^{-1}$$

for all $(\phi, \psi) \in \mathcal{A}$. We say that $f, g \in \mathcal{E}_{n,p}^0$ are \mathcal{A} -equivalent if $g \in \mathcal{A} \cdot f$. A map-germ $f \in \mathcal{E}_{n,p}^0$ is ℓ - \mathcal{A} -determined if every map-germ $g \in \mathcal{E}_{n,p}^0$ with the same ℓ -jet (at 0) as f is \mathcal{A} -equivalent to f . Furthermore, f is *finitely \mathcal{A} -determined* (or \mathcal{A} -finite) if it is ℓ - \mathcal{A} -determined for some $\ell < \infty$. A map-germ is \mathcal{A} -stable if any of its unfoldings is \mathcal{A} -equivalent to the trivial unfolding $f \times 1$. By fundamental

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results of Mather, finite determinacy is equivalent to the finite dimensionality of the normal space

$$(2) \quad N\mathcal{A}_e f := \frac{f^*(\Theta_{\mathbb{C}^p,0})}{tf(\Theta_{\mathbb{C}^n,0}) + f^{-1}(\Theta_{\mathbb{C}^p,0})},$$

and thus (if f is not stable) to $0 \in \mathbb{C}^p$ being an isolated point of instability of f . We set $\mathcal{A}_e\text{-codim}(f) := \dim_{\mathbb{C}} N\mathcal{A}_e f$.

The corank of a map-germ $f \in \mathcal{E}_{n,p}^0$ with $n \leq p$ is defined to be

$$\text{corank } f = n - \text{rank } df(0).$$

Remark 2.1. *a.* There are a few methods to calculate \mathcal{A}_e -codimension for a given finite map-germ and each may have certain disadvantages. Calculating it directly from the definition is not always practical since one has to do it by hand – the normal space is not an $\mathcal{O}_{\mathbb{C}^n,0}$ -module and that makes it difficult to write it into a computer algorithm.

b. Alternatively, one can use J. Damon's theory where he relates \mathcal{A} -equivalence with \mathcal{K}_V equivalence and shows that for a finitely \mathcal{A} -determined map-germ $f \in \mathcal{E}_{n,p}^0$, the normal spaces with respect to \mathcal{A} and \mathcal{K}_V are isomorphic:

$$(3) \quad N\mathcal{A}_e f \cong N\mathcal{K}_{V,eg}$$

where V is the image of a stable unfolding F of f and g is the pull-back map from $(\mathbb{C}^p, 0)$ to the target space of F ([3], see also [19, Theorem 8.1]). The right hand side of (3) can easily be adapted to a computer algebra program (see [1] for examples). However, this procedure requires a long time to complete when the number of parameters for a stable unfolding is too big, as for the examples we study in this article.

Here, we will use the following proposition which provides a shorter and much faster algorithm to calculate \mathcal{A}_e -codimension.

Proposition 2.2 (Proposition 2.1, [17]). *Let h be a defining equation of the image $(X, 0)$ of the finitely \mathcal{A} -determined map-germ $f \in \mathcal{E}_{n,n+1}^0$. Then the evaluation on h defines an isomorphism of $\mathcal{O}_{\mathbb{C}^{n+1},0}$ -modules*

$$(4) \quad N\mathcal{A}_e f \cong \frac{J_h \mathcal{O}_{\mathbb{C}^n,0}}{J_h \mathcal{O}_{X,0}}.$$

Remark 2.3. We understand from the proof of Proposition 2.2 that it is sufficient for f to have a ramification locus of codimension 2 to have the isomorphism in (4).

In what follows, we will denote the right hand side of (4) by N_f .

2.2. Topology of the image. If $f \in \mathcal{E}_{n,p}^0$ is finitely \mathcal{A} -determined then it has an isolated instability at the origin ([15, p. 241], [9]). Moreover, if $(n, n+1)$ are nice dimensions then the image of a stabilisation of f has the homotopy type of wedge of n -spheres ([17, Theorem 1.4]). The number of n -spheres in the wedge is called the image Milnor number and denoted by μ_I .

Remark 2.4. *a.* For map-germs of corank 1 in $\mathcal{E}_{n,n+1}^0$, Goryunov and Mond gave a method to calculate the cohomology of the image of a stable perturbation using alternating cohomology groups of multiple point spaces and that provides a formula for the image Milnor number ([10]). In [12], K. Houston showed that the same formula holds for stable perturbations of map-germs of any corank. See [20] for detailed calculations of μ_I for corank 2 map-germs based on these ideas.

b. For weighted homogeneous map-germs of any corank in $\mathcal{E}_{2,3}^0$, Mond has an ingenious formula for μ_I given in terms of weights and degrees ([16]). In [23], T. Ohmoto improved it to weighted homogenous map-germs in $\mathcal{E}_{3,4}^0$ using characteristic classes and Thom polynomials. That is the formula we will use for our examples in this article.

Clearly, proving the conjecture will provide an alternative method to calculate the image Milnor Number for weighted homogeneous map-germs of any corank. One of the ideas about how to attack the conjecture is based on the relation between \mathcal{A}_e -equivalence and Damon's \mathcal{K}_H -equivalence: The conjecture holds if and only if a particular relative normal space $N\mathcal{K}_{H,e/\mathbb{C}}G$ is a Cohen-Macaulay module ([1], [21]). Recently, J. F. Bobadilla, J.J. Nuño and G. Peñafort proved that it is also equivalent to showing that a *jacobian* module (a relative version of the module $M(f)$ mentioned in Remark 3.2) has the Cohen-Macaulay property ([2]).

3. EXAMPLES

Before we present our examples, we restate the definition for N_f that will help us putting our calculations into a computer algorithm.

Proposition 3.1. *Let $f \in \mathcal{E}_{n,n+1}^0$ be a finite map-germ and let X be its image, defined by an ideal $h \in \mathcal{O}_{\mathbb{C}^{n+1},0}$. Assume that f is a weighted homogeneous map-germ. Then*

$$N_f = \frac{(f^*)^{-1}((f^*J_h)\mathcal{O}_{\mathbb{C}^n,0})}{J_h\mathcal{O}_{\mathbb{C}^{n+1},0}}.$$

If, in addition, the ramification locus of f has codimension 2 then

$$\dim_{\mathbb{C}} N_f = \mathcal{A}_e\text{-codim}(f).$$

Proof. Our argument is based on exploiting the $\mathcal{O}_{\mathbb{C}^{n+1},0}$ -module structure of N_f . The definition of N_f in Proposition 2.2 reads as

$$(5) \quad N_f = \frac{(f^*J_h)\mathcal{O}_{\mathbb{C}^n,0}}{J_h\mathcal{O}_{X,0}}.$$

Let \mathcal{C} be the conductor ideal of $\mathcal{O}_{\mathbb{C}^n,0}$ in $\mathcal{O}_{X,0}$, that is,

$$\mathcal{C} = \{r \in \mathcal{O}_{X,0} \mid r \cdot \mathcal{O}_{\mathbb{C}^n,0} \subset \mathcal{O}_{X,0}\}.$$

We have the following inclusion of the ideals $J_h \subseteq \text{Fitt}_1(f_*\mathcal{O}_{\mathbb{C}^n,0}) \subseteq \mathcal{C}$ (see [22, p.121] for the second inclusion). Notice that since $J_h \subseteq \mathcal{C}$ and

$$\sum_{i=1}^{n+1} \alpha_i \frac{\partial h}{\partial Y_i} \in \mathcal{O}_{X,0}$$

for any $\alpha_i \in \mathcal{O}_{\mathbb{C}^n,0}$, $(f^*J_h)\mathcal{O}_{\mathbb{C}^n,0}$ is an ideal both in $\mathcal{O}_{\mathbb{C}^n,0}$ and in $\mathcal{O}_{X,0}$. Hence, the map

$$\bar{f}^*: \mathcal{O}_{X,0} \rightarrow \mathcal{O}_{\mathbb{C}^n,0}$$

contains $(f^*J_h)\mathcal{O}_{\mathbb{C}^n,0}$ in its image. So, instead of (5), we can take

$$N_f = \frac{(\bar{f}^*)^{-1}((f^*J_h)\mathcal{O}_{\mathbb{C}^n,0})}{J_h\mathcal{O}_{X,0}}.$$

As f is weighted homogeneous, we have $h \in J_h\mathcal{O}_{\mathbb{C}^{n+1},0}$. Therefore,

$$\frac{(\bar{f}^*)^{-1}((f^*J_h)\mathcal{O}_{\mathbb{C}^n,0})}{J_h\mathcal{O}_{X,0}} = \frac{(f^*)^{-1}((f^*J_h)\mathcal{O}_{\mathbb{C}^n,0})}{J_h\mathcal{O}_{\mathbb{C}^{n+1},0}}.$$

Finally, the second part of the statement follows from Remark 2.3. \square

Remark 3.2. The same result, but with a different approach, can also be found in [2]. There, the authors study the kernel $M(f)$ of the epimorphism

$$\frac{\mathcal{C}}{J_h} \rightarrow \frac{\mathcal{C}}{J_h\mathcal{O}_{\mathbb{C}^n,0}}.$$

They show that

$$M(f) = \frac{(f^*)^{-1}((f^*J_h)\mathcal{O}_{\mathbb{C}^n,0})}{J_h\mathcal{O}_{\mathbb{C}^{n+1},0}}$$

and that, for weighted homogeneous map-germs, $M(f) = N_f$ ([2, Proposition 3.3, Proposition 5.1]).

Proposition 3.3. *The map-germ*

$$\begin{aligned} f_1: (\mathbb{C}^3, 0) &\rightarrow (\mathbb{C}^4, 0) \\ (x, y, z) &\mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^7 + x^4z + xz^2 + y^2z) \end{aligned}$$

has \mathcal{A}_e -codimension equal to 18967. It is weighted homogeneous with weights $(1, 2, 3)$ and degrees $(4, 5, 6, 7)$, of corank 3 and satisfy the Mond conjecture.

Proof. Firstly, we have

$$(6) \quad df = \begin{bmatrix} z & 2y & x \\ 5x^4 + y^2 & z + 2xy & y \\ 6x^5 & 3y^2 & 2z \\ 7x^6 + 4x^3z + z^2 & 2yz & x^4 + 2xz + y^2 \end{bmatrix}$$

and $df(0)$ is the zero matrix. Hence f_1 is of corank 3. The ramification locus is defined by 3×3 -minors of df . Its codimension is equal to 2. We check that by the following SINGULAR ([6]) code.

```
ring s=0,(x,y,z),(wp(1,2,3));
ideal f=y2+xz,x5+yz+xy2,x6+y3+z2,x7+x4z+xz2+y2z;
matrix df=jacob(f);
ideal rf=std(minor(df,3));
dim(rf);
//->1.
```

Hence, we can apply Proposition 3.1 to calculate \mathcal{A}_e -codimension of f_1 , i.e. the vector space dimension of N_f . We run the following code to find that.

```
ring t=31991,(X,Y,Z,W),(wp(4,5,6,7)); // the target of f1
ring s=31991,(x,y,z),(wp(1,2,3)); // the domain of f1
map f1=t,y2+xz,x5+yz+xy2,x6+y3+z2,x7+x4z+xz2+y2z;
ideal p=0;
setring t;
ideal h=preimage(s,f1,p); // the ideal defining the image of f1
ideal jh=jacob(h);
setring s;
ideal fjh=f1(jh); // f1*Jh
setring t;
ideal ffjh=preimage(s,f1,fjh); // (f1*)^{-1}f1*Jh
def N=modulo(ffjh,jh);
vdim(std(N));
//->18967.
```

On the other hand, Ohmoto's formula ([23]) for μ_I for weighted homogenous map-germs in $\mathcal{E}_{3,4}^0$ also gives 18967. Therefore, f_1 satisfies the conjecture. \square

Remark 3.4. We carry out our calculations over characteristic 31991 since the computer struggles to give an output over characteristic 0. This choice does not effect the outcome of the code.

Similarly, we can confirm the following examples.

Proposition 3.5. *The map-germ*

$$f_2: (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^4, 0)$$

$$(x, y, z) \mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^9 + x^6z + z^3 + y^3z)$$

has \mathcal{A}_e -codimension equal to 41244. It is weighted homogeneous with weights $(1, 2, 3)$ and degrees $(4, 5, 6, 9)$, of corank 3 and satisfy the Mond conjecture.

Proposition 3.6. *The map-germ*

$$\begin{aligned} f_3: (\mathbb{C}^3, 0) &\rightarrow (\mathbb{C}^4, 0) \\ (x, y, z) &\mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^{13} + x^{10}z + xz^4 + y^5z) \end{aligned}$$

has \mathcal{A}_e -codimension equal to 127295. It is weighted homogeneous with weights $(1, 2, 3)$ and degrees $(4, 5, 6, 13)$, of corank 3 and satisfy the Mond conjecture.

Remark 3.7. It might seem like these three map-germs are parts of a series of map-germs. Ohmoto's formula for the weights $(1, 2, 3)$ and degrees $(4, 5, 6, 2k + 1)$ gives the integer

$$\mu_I = 487k^3 + 576k^2 + 197k + 18 + \frac{4k^3 + 3k^2 + 5k}{6}$$

for all $k \geq 1$. However, the following map-germs are not finitely \mathcal{A} -determined.

$$\begin{aligned} (x, y, z) &\mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^{11} + x^8z + x^2z^3 + y^4z), \\ (x, y, z) &\mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^{15} + x^{12}z + z^5 + y^6z), \\ (x, y, z) &\mapsto (y^2 + xz, x^5 + yz + xy^2, x^6 + y^3 + z^2, x^{17} + x^{14}z + x^2z^5 + y^7z). \end{aligned}$$

Of course, this does not prove that there are not any finitely \mathcal{A} -determined map-germs in $\mathcal{E}_{3,4}^0$ with weights $(1, 2, 3)$ and degrees $(4, 5, 6, 11)$, $(4, 5, 6, 15)$ or $(4, 5, 6, 17)$.

4. OTHER INVARIANTS

In this section we talk about some invariants for the map-germ f_1 introduced in Proposition 3.3. Let us put $f = f_1$ to simplify our notation. The *multiplicity* of f is

$$\text{qf}(f) := \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^3,0}}{f^* \mathfrak{m}_{\mathbb{C}^4,0}} = 17.$$

There exists a presentation of $f_* \mathcal{O}_{\mathbb{C}^3,0}$ of the form

$$0 \rightarrow \mathcal{O}_{\mathbb{C}^4,0}^{17} \xrightarrow{\Lambda} \mathcal{O}_{\mathbb{C}^4,0}^{17} \xrightarrow{\epsilon} f_* \mathcal{O}_{\mathbb{C}^3,0} \rightarrow 0$$

where Λ is a symmetric 17×17 -matrix and ϵ can be chosen as

$$\epsilon = \begin{bmatrix} 1 & x & y & z & x^2 & xy & xz & yz & z^2 & x^3 & x^2y & xyz & x^2z & xz^2 & x^4 & x^3y & x^4y \end{bmatrix}.$$

The k 'th multiple point space M_k on the image is defined by the $(k - 1)$ 'st Fitting ideal $\text{Fitt}_{k-1}(f_* \mathcal{O}_{\mathbb{C}^3,0})$ of $f_* \mathcal{O}_{\mathbb{C}^3,0}$, i.e. the ideal of $(17 - (k - 1)) \times (17 - (k - 1))$ -minors of Λ . For example, M_1 is the image, M_2 is the double point space, ... etc. Let $D_1^k(f)$ be the k 'th multiple point space on the domain with an analytic structure given by $f^* \text{Fitt}_{k-1}(f_* \mathcal{O}_{\mathbb{C}^3,0})$. So, set theoretically, $D_1^k(f) = f^{-1}(M_k)$.

Since f is finitely \mathcal{A} -determined, the multiple point spaces are dimensionally correct, that is,

$$\dim_{\mathbb{C}} D_1^k(f) = 4 - k$$

for $k = 1, 2, 3, 4$. Moreover, no finite map-germ in $\mathcal{E}_{3,4}^0$ admits any *genuine* 5-tuple point.¹ Hence, we are interested in $D_1^k(f)$ only for $k = 2, 3, 4$.

4.1. Triple points.

Lemma 4.1. $\mu(D_1^3(f)) = 168341$.

Proof. We use Greuel's formula for weighted homogeneous space curves given by

$$\mu = \tau - t + 1$$

where t is the Cohen-Macaulay type, i.e. the second Betti number, of the singularity ([11]).

A direct calculation shows that $D_1^3(f)$ is a Cohen-Macaulay space curve with an isolated singularity at the origin. Moreover, it can be defined by 16×16 -minors of a 17×16 -matrix over $\mathcal{O}_{\mathbb{C}^3,0}$.² So, we can use Fröhbis-Krüger's theory ([8]) on matrix singularities to calculate T^1 of $D_1^3(f)$. We run the following code on SINGULAR.

```
LIB "spcurve.lib";
// the ideal of  $D_1^3(f)$  is d31
matrix m31=syz(d31); // a matrix representation of d31
list t31=matrixT1(m31,3);
vdim(std(t31[2])); // the Tjurina number of  $D_1^3(f)$ 
//->168356
CMtype(d31); // Cohen-Macaulay type of  $D_1^3(f)$ 
//-> 16
```

Therefore,

$$\mu(D_1^3(f)) = \tau - t + 1 = 168356 - 16 + 1 = 168341.$$

□

The ramification locus R_f of f_1 is also a space curve with an isolated singularity at the origin. Its matrix is given by (6). So, its Cohen-Macaulay type is 3. We also find that $\tau(R_f) = 127$. Hence,

Lemma 4.2. $\mu(R_f) = 125$.

4.2. Quadruple points. Finding the number of quadruple points of a stable perturbation of f requires a little bit of work. The analytic structure of M_4 only gives us information about the geometrical picture of quadruple points. Whether the vector space dimension of the 3rd Fitting ideal counts the number of quadruple

¹By a genuine k -tuple point, we refer to a point which splits into k distinct points under a stable perturbation.

²In fact, by the Hilbert-Burch theorem, any Cohen-Macaulay variety of codimension 2 can be defined by $r \times r$ -minors of an $(r+1) \times r$ -matrix (see, for example, [7, Theorem 20.15]).

points of a stable perturbation of a map-germ in $\mathcal{E}_{3,4}^0$ is still an open question. For a proof, we need to show that the module

$$M := \frac{\mathcal{O}_{\mathbb{C}^4 \times \mathbb{C}^d, 0}}{\text{Fitt}_3(F_* \mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^d, 0})}$$

satisfy the principle of conservation ([4, Theorem 6.4.7]), where $F \in \mathcal{E}_{3+d,4+d}^0$ is a stable unfolding of f . That is, the stalk of M at 0 is a free $\mathcal{O}_{\mathbb{C}^d, 0}$ -module of finite rank. However, it is a huge task for a computer to conclude such calculation for our example – a stable unfolding of f requires a minimum of 40 parameters. At the moment, we can only conjecture the number of quadruple points.

Conjecture 4.3. *The number of quadruple points is 8970.*

4.3. Double points. For a stable perturbation f_t of f , $D_1^2(f_t)$ has the homotopy type of a wedge of 2-spheres (see [20, Remark 1.1 (2)]). We would also like to calculate the number of spheres in the wedge using the methods explained in [20]. However, due to computer memory restrictions, we have to leave this question to another study.

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